

The $e^+e^- \rightarrow P_1P_2\gamma$ processes close to the Φ peak: toward a model-independent analysis

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ABSTRACT: We discuss the general decomposition and possible general parameterizations of the processes $e^+e^- \rightarrow \gamma^* \rightarrow P_1P_2\gamma$, where $P_1P_2 = \pi^0\pi^0$, $\pi^0\eta$, or $\pi^+\pi^-$, for $\sqrt{s} \approx M_\Phi$. Particular attention is devoted to the amplitude where the two pseudoscalar mesons are in a $J^{CP} = 0^{++}$ state, where we propose a general parameterization which should help to shed light on the nature of light scalar mesons.

KEYWORDS: Chiral Lagrangians, QCD, Phenomenological Models, Electromagnetic Processes and Properties.

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1. Introduction

As widely discussed in the literature, radiative Φ decays, such as $\Phi \rightarrow f_0\gamma \rightarrow \pi\pi\gamma$ or $\Phi \rightarrow a_0\gamma \rightarrow \pi\eta\gamma$, are one of the primary sources of information about the interesting and still controversial sector of light scalar mesons (see e.g. refs. [1–3] and references therein). In principle, the large amount of data collected at Φ factories should allow to study these processes with excellent accuracy. However, it must be realized that these processes are only one of the components of the basic Φ -factory observables, namely the $e^+e^- \rightarrow P_1P_2\gamma$ cross sections (where $P_1P_2 = \pi^0\pi^0, \pi^0\eta$, or $\pi^+\pi^-$). An accurate and possibly model-independent description of all the components of these reactions is necessary in order to extract reliable information about the scalar sector of QCD.

The purpose of the present paper is twofold. First, we discuss how to isolate the interesting scalar amplitude from the other contributions to the cross section. Second, we present a general parameterization of the scalar amplitude which should allow a model-independent determination of basic parameters such as masses, widths and couplings of f_0 and a_0 mesons.

The paper is organized as follows. In section 2 we discuss the general decomposition the cross sections in terms of gauge-invariant tensors. We start from the simpler case of neutral final states, $|\pi^0\pi^0\gamma\rangle$ and $|\pi^0\eta\gamma\rangle$. We then generalize to the $|\pi^+\pi^-\gamma\rangle$ case, where initial- and final-state radiation represent a serious background. In section 3 we analyse the contributions to the invariant amplitudes due to the exchange of vector mesons ($e^+e^- \rightarrow VP_{1(2)} \rightarrow P_1P_2\gamma$), which represent an irreducible physical background for the scalar amplitude. Finally, in section 4 we present a general decomposition of the scalar

amplitude ($e^+e^- \rightarrow S\gamma \rightarrow P_1P_2\gamma$) which takes into account the narrow-width structure of f_0 and a_0 at high $M_{P_1P_2}$, and the constraints of unitarity and chiral symmetry at low $M_{P_1P_2}$. The complete procedure we propose for the fit of the cross sections is summarized in the last section.

2. General decomposition of the cross sections

2.1 Neutral final states

The generic matrix element for the transition $e^+e^- \rightarrow P_1^0P_2^0\gamma$ ($P_{1,2} = \pi^0$ or η) can be written as

$$\mathcal{M} [e^+(p_+)e^-(p_-) \rightarrow P_1^0(p_1)P_2^0(p_2)\gamma(\varepsilon, k)] = \frac{e}{s}\bar{v}(p_+)\gamma_\mu u(p_-)T^{\mu\nu}\varepsilon_\nu, \quad (2.1)$$

where $s = (p_+ + p_-)^2 \equiv P^2$. The constraints of gauge invariance imply

$$P_\mu T^{\mu\nu} = k_\nu T^{\mu\nu} = 0. \quad (2.2)$$

Using this notation, the differential cross section with unpolarized beams becomes

$$\begin{aligned} d\sigma(e^+e^- \rightarrow P_1^0P_2^0\gamma) &= \frac{1}{8s}C_{12} \sum_{\text{spins}} |\mathcal{M}|^2 d\Phi \\ &= \frac{8\pi\alpha}{s^3}C_{12} \left[p_+^\mu p_-^\nu + p_-^\mu p_+^\nu - \frac{s}{2}g^{\mu\nu} \right] \left[-\frac{1}{4}g^{\rho\sigma}T_{\mu\sigma}T_{\nu\rho}^* \right] d\Phi, \end{aligned} \quad (2.3)$$

where

$$d\Phi = \frac{d^3k}{(2\pi)^3 2E_\gamma} \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2 - k) \quad (2.4)$$

and C_{12} takes into account the 1/2 factor in case of identical particles: $C_{12} = 1$ for $\{P_1^0, P_2^0\} \equiv \{\eta, \pi^0\}$; $C_{12} = 1/2$ for $\{P_1^0, P_2^0\} = \{\pi^0, \pi^0\}$.

As a result of the gauge-invariance conditions, in the case of a real photon in the final state we can decompose $T^{\mu\nu}$ as the sum of three independent structures:

$$T_{\mu\nu} = \frac{4}{s} \left[A_1 L_{\mu\nu}^{(1)} + A_2 L_{\mu\nu}^{(2)} + A_3 L_{\mu\nu}^{(3)} \right], \quad (2.5)$$

with

$$\begin{aligned} L_{\mu\nu}^{(1)} &= (k \cdot P)g_{\mu\nu} - k_\mu P_\nu, \\ L_{\mu\nu}^{(2)} &= \frac{4}{s} \{ (P \cdot q) [(k \cdot q)g_{\mu\nu} - k_\mu q_\nu] + (k \cdot P)q_\mu q_\nu - (k \cdot q)q_\mu P_\nu \}, \\ L_{\mu\nu}^{(3)} &= \frac{4}{s} [(k \cdot q)(P^2 g_{\mu\nu} - P_\mu P_\nu) + (k \cdot P)P_\mu q_\nu - P^2 k_\mu q_\nu], \quad q = \frac{1}{2}(p_1 - p_2). \end{aligned}$$

The normalization of $T_{\mu\nu}$ and the $L_{\mu\nu}^{(i)}$ has been chosen in order to maximize the contact with ref. [4]: $L_{\mu\nu}^{(1)}$ and $L_{\mu\nu}^{(3)}$ are identical to the corresponding expressions of ref. [4], while $L_{\mu\nu}^{(2)}$ coincides with the corresponding tensor of ref. [4] only in the case of identical particles.

The dimensionless form factors A_i are determined by the specific dynamical model. For instance, the scalar amplitude $e^+e^- \rightarrow S\gamma \rightarrow P_1P_2\gamma$ induces a non-vanishing contribution only to A_1 , while the transition $e^+e^- \rightarrow VP_{1(2)} \rightarrow P_1P_2\gamma$ leads to non-vanishing contributions to all the form factors.

The phase space element can be re-written as

$$d\Phi = \frac{1}{8(2\pi)^4} dE_1 dE_\gamma d\Omega_b = \frac{1}{8(2\pi)^4} dE_1 dE_2 d\Omega_b, \quad (2.6)$$

where $E_{\gamma,1,2}$ are the energies in the c.o.m. frame and Ω_b denotes the solid angle of the beam axis with respect to the decay plane.¹ Performing the angular integration we find

$$\begin{aligned} F(x, x_1, x_2) &= \frac{1}{2\pi s} \int d\Omega_b \left[-\frac{1}{4} g^{\rho\sigma} T_{\mu\sigma} T_{\nu\rho}^* \right] \left[p_+^\mu p_-^\nu + p_-^\mu p_+^\nu - \frac{s}{2} g^{\mu\nu} \right] \\ &= a_{11}|A_1|^2 + a_{22}|A_2|^2 + a_{33}|A_3|^2 + a_{12}(A_1 A_2^* + A_1^* A_2) \\ &\quad + a_{13}(A_1 A_3^* + A_1^* A_3) + a_{23}(A_2 A_3^* + A_2^* A_3), \end{aligned} \quad (2.7)$$

where the a_{ij} coefficients are given by

$$\begin{aligned} a_{11} &= \frac{4}{3} x^2 \\ a_{12} &= \frac{2}{3} [(x_1 - x_2)^2 + x^2(\sigma - 1 + x) - 2\delta(x_1 - x_2) + \delta^2] \\ a_{13} &= \frac{8}{3} x(x_1 - x_2 - \delta) \\ a_{22} &= \frac{1}{3} \{ (x_1 - x_2)^4 + 2x^2(\sigma - 1 + x)^2 + 2(x_1 - x_2)^2(1 - x)(\sigma - 1 + x) \\ &\quad - \delta(x_1 - x_2) [2(x_1 - x_2)^2 + 2(\sigma - 1 + x)(x_1 + x_2)] \\ &\quad + \delta^2 [(x_1 - x_2)^2 + 2(\sigma - 1 + x)] \} \\ a_{23} &= \frac{4}{3} (x_1 - x_2) [(x_1 - x_2)^2 - 2\delta(x_1 - x_2) + \delta^2] \\ a_{33} &= \frac{8}{3} [(x_1 - x_2)^2(1 + x) - x^2(\sigma - 1 + x) - \delta(x_1 - x_2)(x + 2) + \delta^2] \end{aligned} \quad (2.8)$$

in terms of the adimensional variables

$$\begin{aligned} x_i &= \frac{2E_i}{\sqrt{s}} & x &= \frac{2E_\gamma}{\sqrt{s}} = 2 - x_1 - x_2 \\ \sigma &= \frac{2(M_1^2 + M_2^2)}{s} & \delta &= \frac{2(M_1^2 - M_2^2)}{s}. \end{aligned} \quad (2.9)$$

In the limit $\delta \rightarrow 0$ the a_{ij} coefficients in eq. (2.8) coincide with the C_{ij} of ref. [4], with the exception of a_{12} , which is reported incorrectly in ref. [4].

The general expression for the total cross-section is then given by

$$\begin{aligned} d\sigma(e^+e^- \rightarrow P_1^0 P_2^0 \gamma) &= \frac{\alpha}{32\pi^2 s} C_{12} F(2 - x_1 - x_2, x_1, x_2) dx_1 dx_2 \\ &= \frac{\alpha}{32\pi^2 s} C_{12} F(x, x_1, 2 - x - x_1) dx dx_1. \end{aligned} \quad (2.10)$$

¹ The element of the solid angle can be expressed as $d\Omega_b = d\cos\theta_\gamma d\phi$, where θ_γ is the angle between photon and e^+ momenta, and ϕ the orthogonal angle with respect to the decay plane. The latter leads to a trivial integration in the $|A_1|^2$ term, but is non-trivial for the other contributions.

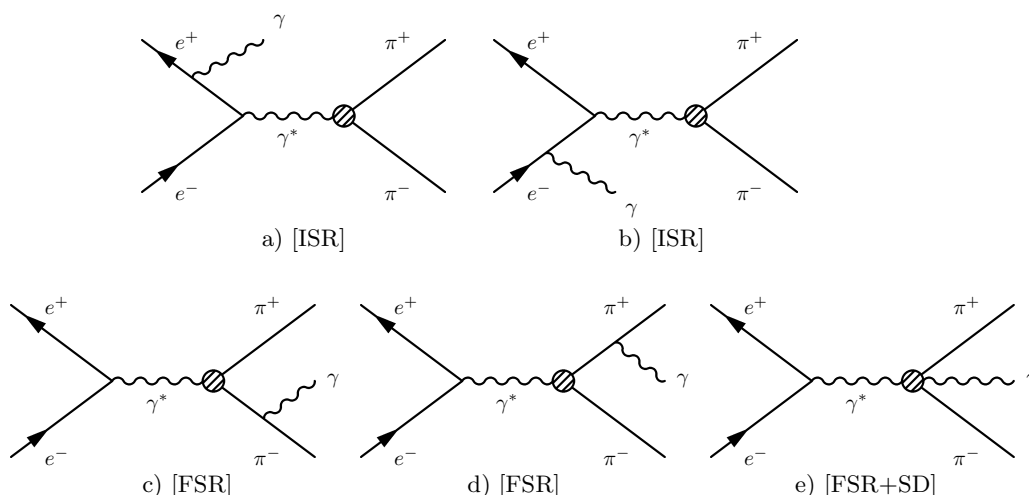


Figure 1: Tree-level diagrams for ISR, FSR and SD contributions to $e^+e^- \rightarrow \pi^+\pi^-\gamma$. The SD amplitude corresponds to the $\mathcal{O}(E_\gamma)$ gauge-invariant terms in the diagram e).

2.2 Initial- and final-state radiation and generalization to the $e^+e^- \rightarrow \pi^+\pi^-\gamma$ case

At $\mathcal{O}(\alpha^2)$ we can distinguish three basic components to the processes $e^+e^- \rightarrow \pi^+\pi^-\gamma$: initial-state radiation (ISR), final-state radiation (FSR) and the irreducible structure-dependent (SD) amplitude which vanishes in the $E_\gamma \rightarrow 0$ limit (see figure 1). The latter, which is identical to the neutral case discussed before, is the only contribution sensitive to scalar-meson dynamics.

By construction, ISR and FSR amplitudes can be fully described in terms of the electromagnetic form factor of the pion and are known with good accuracy² (for an extensive discussion applied to the Φ -factory case see e.g. ref. [6, 7]). The $|\pi^+\pi^- \rangle$ state produced by ISR has opposite transformation properties under parity and charge conjugation with respect to those produced by FSR and SD. For this reason, as long as the kinematical cuts applied on the cross section are symmetric under the exchange $\pi^+ \leftrightarrow \pi^-$, we can neglect the interference of the ISR amplitude with the other two contributions. On the other hand, the FSR amplitude, which can be decomposed according to eq. (2.5), does interfere with the SD terms:

$$d\sigma(e^+e^- \rightarrow \pi^+\pi^-\gamma) = d\sigma_{\text{ISR}} + d\sigma_{\text{FSR}} + d\sigma_{\text{SD}} + d\sigma_{\text{FSR+SD}}^{\text{Interf}}. \quad (2.11)$$

The form factors of the FSR contribution, which are singular in the $x \rightarrow 0$ limit, can be written as

$$A_1^{\text{FSR}} = -\frac{4\pi\alpha x F_{\pi\pi}(s)}{x^2 - (x_1 - x_2)^2}, \quad A_2^{\text{FSR}} = \frac{8\pi\alpha F_{\pi\pi}(s)}{x^2 - (x_1 - x_2)^2}, \quad A_3^{\text{FSR}} = 0, \quad (2.12)$$

² We denote here as FSR amplitude only the gauge-invariant part of the amplitude which can unambiguously related to the on-shell non-radiative $e^+e^- \rightarrow \pi^+\pi^-$ processes, according to Low's theorem [5].

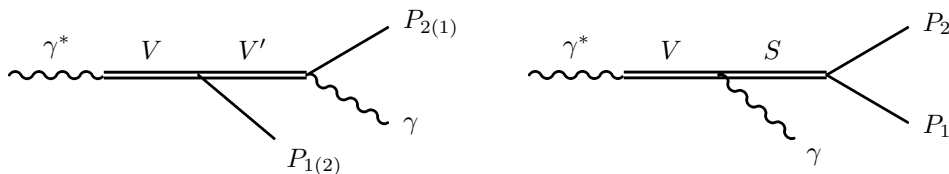


Figure 2: Leading contributions to the structure-dependent amplitude: vector-meson exchange (left) and scalar (right) contributions.

where $F_{\pi\pi}(s)$ denotes the pion electromagnetic form factor (with the standard normalization $F_{\pi\pi}(0) = 1$). The last three terms in eq. (2.11) can thus be described by a generic expression of the type in eq. (2.10), where the form factors are obtained by summing the singular A_i^{FSR} to the regular terms of the SD amplitude.

So far we have analysed only the $\mathcal{O}(\alpha^2)$ contributions to the cross sections, or the leading contributions in the case of a single detected photon in the final state. As is well known, the physical cross sections are obtained by the convolution of these leading expressions with appropriate radiation functions [8] which take into account the effect of the undetected ISR (both for charged and neutral final states) and the undetected FSR (for the charged final state only).

3. The vector-meson exchange contribution

In a good fraction of the allowed kinematical region, a sizable contribution to the structure-dependent amplitude is induced by the vector-meson exchange process in figure 2 (left).

The starting point to evaluate this contribution are the effective couplings $g_{PV'}^V$ and $g_{P\gamma}^V$ (with dimension 1/energy), defined by

$$\begin{aligned} \mathcal{M}[V(\tilde{\varepsilon}, P) \rightarrow P(q)\gamma(\varepsilon, k)] &= eg_{P\gamma}^V \epsilon^{\mu\nu\rho\sigma} \tilde{\varepsilon}_\mu \varepsilon_\nu q_\rho k_\sigma, \\ \mathcal{M}[V(\tilde{\varepsilon}, P) \rightarrow P(q)V'(\varepsilon, k)] &= g_{PV'}^V \epsilon^{\mu\nu\rho\sigma} \tilde{\varepsilon}_\mu \varepsilon_\nu q_\rho k_\sigma, \end{aligned} \quad (3.1)$$

and the adimensional vector-meson electromagnetic couplings, F_V , defined by

$$\mathcal{M}[V(\varepsilon) \rightarrow e^+e^-] = \frac{e}{F_V} \varepsilon_\mu \bar{u}(p_-)\gamma^\mu v(p_+). \quad (3.2)$$

In terms of these couplings, the vector-meson exchange process in figure 2 give rise to the following contributions to the form factors:

$$\begin{aligned} A_1^{\text{vect}} &= -\frac{1}{4} \left(-1 + \frac{3}{2}x + \sigma \right) [g(x_1) + g(x_2)] + \frac{1}{4} \left(x_1 - x_2 + \frac{1}{2}\delta \right) [g(x_1) - g(x_2)] \\ A_2^{\text{vect}} &= \frac{1}{4} [g(x_1) + g(x_2)] \\ A_3^{\text{vect}} &= -\frac{1}{8} [g(x_1) - g(x_2)] \end{aligned} \quad (3.3)$$

where

$$g(y) = \sum_{V, V'=\rho, \omega, \Phi, \rho', \omega', \dots} \frac{eg_{PV'}^V g_{P\gamma}^{V'}}{4F_V} \frac{s^2 M_V^2}{D_V(s) D_{V'}[(1-y)s]} \quad (3.4)$$

F_V	$g_{P\gamma}^V$ [GeV $^{-1}$]	$g_{PV'}^V = g_{PV}^{V'}$ [GeV $^{-1}$]
$ F_\rho = 16.6 \pm 0.2$	$ g_{\pi\gamma}^\rho = 0.79 \pm 0.10$ $ g_{\eta\gamma}^\rho = 1.56 \pm 0.08$	$ g_{\pi\rho}^\omega = 12 \pm 2$ $ g_{\pi\rho}^\Phi = 1.3 \pm 0.2$
$ F_\omega = 55.9 \pm 0.5$	$ g_{\pi\gamma}^\omega = 2.38 \pm 0.04$ $ g_{\eta\gamma}^\omega = 0.46 \pm 0.02$	$ g_{\pi\omega}^\Phi = (4.1 \pm 0.5) \times 10^{-2}$ $ g_{\eta\omega}^\Phi = 1.1 \pm 0.2$
$ F_\Phi = 44.4 \pm 0.4$	$ g_{\pi\gamma}^\Phi = 0.132 \pm 0.004$ $ g_{\eta\gamma}^\Phi = 0.691 \pm 0.008$	$ g_{\eta\Phi}^\Phi = 2 g_{\eta\rho}^\rho = 2 g_{\eta\omega}^\omega = 7.0 \pm 1.5$

Table 1: Reference values for the modulus of the effective vector-meson couplings defined in eqs. (3.1)–(3.2).

and

$$D_X(q^2) = s - M_X^2 + iM_X\Gamma_X. \quad (3.5)$$

In the limit of identical particles in the final state, these results are fully consistent with those of ref. [4]. As discussed in ref. [4], a natural improvement of the above expressions is obtained by the replacement of the constant Γ_X with appropriate energy-dependent widths, taking into account the velocity factors of the dominant final states [9].

In table 1 we report the current estimates for the most relevant set of F_V , $g_{P\gamma}^V$, and $g_{PV'}^V$ couplings. The results for F_V and $g_{P\gamma}^V$ have been determined by means of the relations

$$\Gamma(V \rightarrow e^+e^-) = \frac{\alpha M_V}{3|F_V|^2}, \quad \Gamma(V \rightarrow P\gamma) = \frac{\alpha |g_{P\gamma}^V|^2}{3} \left[\frac{M_V^2 - M_P^2}{2M_V} \right]^3, \quad (3.6)$$

using the experimental values of $\Gamma(V \rightarrow e^+e^-)$ and $\Gamma(V \rightarrow P\gamma)$ in [10]. The $g_{PV'}^V$ have been determined theoretically,³ with the exception of $g_{\pi\omega}^\Phi$, which has been determined directly from the experimental value of $\Gamma(\Phi \rightarrow \omega\pi)$ in [10]. We stress that the results reported in table 1 should be considered only as rough reference values, or as natural starting point for a fit of the cross sections. The high-statistics data on the $e^+e^- \rightarrow P_1P_2\gamma$ reactions at a Φ factory should allow to determine these couplings (or at least some combinations of them) with much higher accuracy.

4. The scalar amplitude

We are now ready to analyse the scalar amplitude or, more precisely, the contributions to the form factor A_1 not described by vector mesons and/or FSR.

The contribution to A_1 induced by the exchange of a single vector and a single scalar resonance, as shown in figure 2 (right), is:

$$A_1(e^+e^- \rightarrow V \rightarrow S\gamma \rightarrow P_1P_2\gamma) = \frac{eg_{12}^S g_{S\gamma}^V}{4F_V} \frac{sM_V^2}{D_V(s)D_S[(1-x)s]} \quad (4.1)$$

³ The $g_{PV'}^V$ have been determined by: i) assuming a simple effective Lagrangian of the type $\mathcal{L} = g \text{tr}(\{V, V'\}P)$, where V , V' , and P are 3×3 matrices in flavour space; ii) fixing the Φ - ω mixing angle from the F_V values; iii) enforcing the vector-meson dominance relation $g_{P\gamma}^V = \sum_{V'} g_{PV'}^V / (eF_{V'})$.

where the couplings g_{12}^S and $g_{S\gamma}^V$ are defined by

$$\Gamma(V \rightarrow S\gamma) = \frac{\alpha |g_{S\gamma}^V|^2}{3} \left[\frac{M_V^2 - M_S^2}{2M_V} \right]^3, \quad (4.2)$$

$$\Gamma(S \rightarrow P_1 P_2) = \frac{|g_{12}^S|^2 p_{12}^*(M_S^2)}{8\pi M_S^2}, \quad (4.3)$$

$$p_{12}^*(M^2) = \frac{[M^2 - (M_1 - M_2)^2]^{1/2} [M^2 - (M_1 + M_2)^2]^{1/2}}{2M}.$$

If we could consider only this resonant contribution, the total cross-section would assume the following form

$$\begin{aligned} d\sigma_{\text{Scalar}} &= \frac{2\alpha}{3\pi^2 s^3} C_{12} |A_1(e^+e^- \rightarrow V \rightarrow S\gamma \rightarrow P_1 P_2\gamma)|^2 E_\gamma^2 dE_\gamma dE_1 \\ &= \frac{C_{12}}{4\pi^2 s} BW_V(s) \frac{\Gamma(V \rightarrow e^+e^-)}{M_V \Gamma_V^2} \left| \frac{eg_{12}^S g_{S\gamma}^V}{D_S(s_{12})} \right|^2 \frac{M_V^2 E_\gamma^3 p_{12}^*(s_{12})}{\sqrt{s_{12}}} dE_\gamma, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} s_{12} &= s - 2\sqrt{s}E_\gamma = (1-x)s, \\ BW_V(s) &= \frac{M_V^2 \Gamma_V^2}{|D_V(s)|^2} = \frac{M_V^2 \Gamma_V^2}{(M_V^2 - s)^2 + M_V^2 \Gamma_V^2}. \end{aligned} \quad (4.5)$$

Note that the E_γ^3 factor in (4.4), which is dictated by gauge-invariance according to the general decomposition (2.8), implies a sizable distortion from a standard Breit-Wigner shape for resonances close to the end of the phase space, such as $f_0(980)$ and $a_0(980)$.

The simplified expression (4.4) is often used in the literature to describe the contributions of the narrow resonances $f_0(980)$ and $a_0(980)$. However, a coherent description of all the amplitudes contributing to the physical processes requires a more refined treatment. First, in order to compute the total cross section we need to consider the general expression in eq. (2.10), with coherent sum of all contributions to the A_i :

$$A_1^{\text{full}} = A_1^{\text{FSR}} + A_1^{\text{vect}} + A_1^{\text{scal}}, \quad A_2^{\text{full}} = A_2^{\text{FSR}} + A_2^{\text{vect}}, \quad A_3^{\text{full}} = A_3^{\text{vect}}, \quad (4.6)$$

with A_i^{FSR} ($\pi^+\pi^-$ case only) and A_i^{vect} given in eq. (2.12) and eq. (3.3), respectively. Second, the expression of A_1^{scal} can involve several resonances and, possibly, also non-resonant backgrounds.

Since $f_0(980)$ and $a_0(980)$ are the only two narrow scalar resonances with mass below M_Φ , a convenient parameterization for A_1^{scal} is

$$A_1^{\text{scal}} = \sum_{V=\rho,\omega,\phi; S=f_0,a_0} \frac{e}{4F_V} \frac{sM_V^2}{D_V(s)} \left[\frac{g_{12}^S g_{S\gamma}^V}{D_S(s_{12})} + R_{12}^V(s_{12}) \right]. \quad (4.7)$$

Here $R_{12}^V(s_{12})$ denotes a non-resonant term which, in absence of re-scattering effects, can be expanded as a regular series in powers of $(s_{12} - M_S^2)$. The situation becomes particularly simple in the exact isospin limit, where a single narrow resonance can contribute to each

channel: the $f_0(I = 0)$ in the two $|\pi\pi\rangle$ cases and the $a_0(I = 1)$ in the $|\pi\eta\rangle$ one. In this limit, at fixed values of s and neglecting re-scattering effects, we expect a structure of the type

$$A_1^{\text{scal}} \propto \frac{g_{\text{eff}}^S}{s_{12} - M_S^2 + iM_S\Gamma_S} + \frac{\alpha_0}{M_\Phi^2} + \frac{\alpha_1}{M_\Phi^4}(s_{12} - M_S^2) + \mathcal{O}[(s_{12} - M_S^2)^2], \quad (4.8)$$

with a different set of effective couplings for each channel. As we will discuss in the following, this structure can be systematically improved in order to take into account the elastic re-scattering phases of the two pseudoscalar mesons.

It is worth to stress that a sizable non-resonant term is phenomenologically required from data, at least in the $|\pi\pi\rangle$ channels. Indeed, if we only retain the pole term in eq. (4.8), the E_γ^3 factor in the cross section — see eq. (4.4) — implies a too large result at low $s_{\pi\pi}$ compared to observations (see e.g. ref.[11, 12]): experimental data clearly indicate that at low $s_{\pi\pi}$ the contribution of the $f_0(980)$ is partially compensated by other contributions.

In the $|\pi\pi\rangle$ channels one can consider an alternative parameterization where, in addition to the narrow $f_0(980)$, also the broad $f_0(600)$ is included by means of an appropriate complex propagator in the $I = 0$ channel.⁴ In this case the $f_0(600)$ could be responsible for the partial cancellation of the $f_0(980)$ contribution at low $s_{\pi\pi}$, with a minor role played by the non-resonant term. In principle, precise enough experimental data on the $e^+e^- \rightarrow \pi\pi\gamma$ cross sections should be able to distinguish the case of an explicit pole structure for the $f_0(600)$ from a pure polynomial term. However, the broad nature of the $f_0(600)$ makes this distinction very difficult, even with the high statistics available at a Φ factory.

4.1 Re-scattering phases for $|\pi\pi\rangle$ final states

As anticipated, the parameterization (4.8) can be improved in order to take into account the absorptive parts generated by elastic re-scattering. We illustrate here how this improvement can be implemented in the $|\pi\pi\rangle$ case, where this effect is more relevant and the information about elastic re-scattering at low $s_{\pi\pi}$ is very precise. For simplicity, we consider only Φ -mediated contributions and we include only the $f_0(980)$ as explicit pole structure.

Neglecting non- Φ contributions, we can decompose the scalar form factors for charged and neutral $|\pi\pi\rangle$ final states as

$$\begin{aligned} A_1^{\text{scal}}(\pi^+\pi^-) &= \frac{e}{4F_\Phi} \frac{sM_\Phi^2}{D_\Phi(s)} \frac{F_2^{\text{scal}}(s_{\pi\pi}) + \sqrt{2}F_0^{\text{scal}}(s_{\pi\pi})}{\sqrt{3}}, \\ A_1^{\text{scal}}(\pi^0\pi^0) &= \frac{e}{4F_\Phi} \frac{sM_\Phi^2}{D_\Phi(s)} \frac{\sqrt{2}F_2^{\text{scal}}(s_{\pi\pi}) - F_0^{\text{scal}}(s_{\pi\pi})}{\sqrt{3}}, \end{aligned} \quad (4.9)$$

⁴ The existence of a pole in the S -wave, $I = 0$, $\pi\pi$ scattering amplitude —corresponding to the $f_0(600)$ — is not under doubt [14]. However, this pole is very far from the real axis and quite close to the $\pi\pi$ threshold [14]. As a consequence, its contribution to the amplitude (4.7) is not necessarily well described by a simple complex propagator as in the $f_0(980)$ case.

where $s_{\pi\pi} = (1-x)s$ and the reduced $F_{0,2}^{\text{scal}}$ correspond the $I = 0, 2$ isospin combinations. If we include only the $f_0(980)$ as explicit pole structure, we then have⁵

$$F_0^{\text{scal}}(s_{\pi\pi}) = \frac{g_{\pi\pi}^{f_0} g_{f_0\gamma}^{\Phi}}{s_{\pi\pi} - M_{f_0}^2 + i\sqrt{s_{\pi\pi}} \Gamma_{f_0}(s_{\pi\pi})} + R_0(s_{\pi\pi}),$$

$$F_2^{\text{scal}}(s_{\pi\pi}) = R_2(s_{\pi\pi}). \quad (4.10)$$

For $s_{\pi\pi}$ close to the $f_0(980)$ pole, the leading elastic and inelastic re-scattering effects are automatically included in the scalar resonance propagator [9]. In particular, considering only two-body intermediate states ($\pi\pi$ and $K\bar{K}$) and defining

$$\Sigma_{f_0}(s) = i\sqrt{s}\Gamma_{f_0}(s) \quad \text{and} \quad v_i(s) = (s/4 - M_i^2)^{1/2}, \quad (4.11)$$

the effective energy-dependent width assumes the form

$$4M_{\pi}^2 \leq s < 4M_{K^{\pm}}^2 \quad \Sigma_{f_0}(s) = \frac{1}{8\pi\sqrt{s}} \left\{ -(g_{KK}^{f_0})^2 \left[|v_{K^+}(s)| + |v_{K^0}(s)| \right] + i(g_{\pi\pi}^{f_0})^2 v_{\pi}(s) \right\},$$

$$4M_{K^{\pm}}^2 \leq s < 4M_{K^0}^2 \quad \Sigma_{f_0}(s) = \frac{1}{8\pi\sqrt{s}} \left\{ (g_{KK}^{f_0})^2 \left[-|v_{K^0}(s)| + i v_{K^+}(s) \right] + i(g_{\pi\pi}^{f_0})^2 v_{\pi}(s) \right\},$$

$$s \geq 4M_{K^0}^2 \quad \Sigma_{f_0}(s) = \frac{i}{8\pi\sqrt{s}} \left\{ (g_{KK}^{f_0})^2 \left[v_{K^+}(s) + v_{K^0}(s) \right] + (g_{\pi\pi}^{f_0})^2 v_{\pi}(s) \right\}. \quad (4.12)$$

The key observation which allows to determine the absorptive parts of the form factors in the low $s_{\pi\pi}$ region is the Fermi-Watson theorem [13]. This implies that below the inelastic threshold the phases of the two $F_I^{\text{scal}}(s_{\pi\pi})$ coincide with the $\pi\pi$ S -wave elastic phases.

The low-energy structure of $\pi\pi$ phase shifts is known very precisely [15]. In the S -wave channels we can write

$$\tan[\delta_I(s)] \approx \frac{2v_{\pi}(s)}{\sqrt{s}} \left(a_I + b_I \frac{s - s_0}{s_0} \right),$$

with $s_0 = 4M_{\pi}^2$, and the a_I and b_I reported in table 2.

	a_I	b_I
$I = 0$	0.220 ± 0.005	0.275 ± 0.009
$I = 2$	0.045 ± 0.001	0.081 ± 0.002

Table 2: Numerical values for the low-energy parameters of S -wave $\pi\pi$ phases from ref. [15].

In the $I = 2$ case, where the resonance term is absent, the Fermi-Watson constraint applies directly to the non-resonant term $R_2(s_{\pi\pi})$. Generalizing the polynomial expansion in eq. (4.8), this implies

$$R_2(s_{\pi\pi}) = \left(\frac{\gamma_0}{M_{\Phi}^2} + \frac{\gamma_1}{M_{\Phi}^4} (s_{\pi\pi} - M_{f_0}^2) + \mathcal{O}[(s_{\pi\pi} - M_{f_0}^2)^2] \right) e^{i\delta_2(s_{\pi\pi})}, \quad (4.13)$$

where the γ_i are real parameters, as implicitly assumed for all the effective couplings so far introduced ($g_{S\gamma}^V, g_{P\gamma}^V, g_{12}^S \dots$).

The condition to be imposed on the $I = 0$ term is slightly more complicated since we cannot ignore the phase shift induced by the f_0 propagator. Given that below the inelastic threshold all phase shifts are proportional to the pion velocity $v_{\pi}(s_{\pi\pi})$, a convenient decomposition for $R_0(s_{\pi\pi})$ is

$$R_0(s_{\pi\pi}) = \frac{\alpha_0}{M_{\Phi}^2} e^{i\beta_0 \frac{v_{\pi}(s_{\pi\pi})}{M_{\Phi}}} + \frac{\alpha_1}{M_{\Phi}^4} e^{i\beta_1 \frac{v_{\pi}(s_{\pi\pi})}{M_{\Phi}}} (s_{\pi\pi} - M_{f_0}^2) + \mathcal{O}[(s_{\pi\pi} - M_{f_0}^2)^2], \quad (4.14)$$

⁵ The effective coupling $g_{\pi\pi}^{f_0}$ denotes the coupling of the $f_0(980)$ to the two-pion state with $I = 0$. Correspondingly, the decay width appearing in the definition (4.3) must be interpreted as $\Gamma[f_0 \rightarrow (\pi\pi)_{I=0}] = \Gamma(f_0 \rightarrow \pi^+\pi^-) + \Gamma(f_0 \rightarrow \pi^0\pi^0)$.

where α_i and β_i are real parameters. Imposing the Fermi-Watson constraint allows to determine the β_i in terms of the α_i and the parameters of $\delta_0(s_{\pi\pi})$. Indeed, expanding the phase of the full form factor in powers of $(s_{\pi\pi} - s_0)$, up to second order, leads to the following two conditions:

$$\beta_0 = \frac{M_\Phi^3}{\alpha_0} [\xi(s_0) - (s_0 - M_{f_0}^2)\xi'(s_0)] , \quad \beta_1 = \frac{M_\Phi^5}{\alpha_1} \xi'(s_0) , \quad (4.15)$$

where

$$\begin{aligned} \xi(s) = & \frac{(g_{\pi\pi}^{f_0})^3 g_{f\gamma}^\phi / (8\pi\sqrt{s})}{(s - M_{f_0}^2 + \text{Re}[\Sigma_{f_0}(s)])^2 + \text{Im}[\Sigma_{f_0}(s)]^2} + \frac{2}{\sqrt{s}} \left(a_0^0 + b_0^0 \frac{s - s_0}{s_0} \right) \times \\ & \times \left[\frac{g_{\pi\pi}^{f_0} g_{f\gamma}^\phi (s - M_{f_0}^2 + \text{Re}[\Sigma_{f_0}(s)])}{(s - M_{f_0}^2 + \text{Re}[\Sigma_{f_0}(s)])^2 + \text{Im}[\Sigma_{f_0}(s)]^2} + \frac{\alpha_0}{M_\Phi^2} + \frac{\alpha_1}{M_\Phi^4} (s - M_{f_0}^2) \right] \end{aligned} \quad (4.16)$$

and $\xi'(s) = d\xi/ds$.

Proceeding in a similar way, this method can be generalized to include non- Φ contributions and/or the explicit pole structure of the $f_0(600)$ and/or additional polynomial terms in the Taylor expansion.

5. Summary

The procedure we propose for a general un-biased analysis of the $e^+e^- \rightarrow P_1 P_2 \gamma$ cross sections can be summarized as follows:

- According to eqs. (2.7)–(2.10), the Born (single-photon) cross sections are expressed in terms of the three Lorentz-invariant form factors A_{1-3} .
- The three form factors are decomposed as in (4.6) in terms of a FSR component ($A_{1,2}^{\text{FSR}}$) and a two leading SD components (A_1^{scal} and A_1^{vect}). The FSR component, which is present only in the $|\pi^+\pi^-\gamma\rangle$ case, is fully determined by the electromagnetic pion form factor.
- The vector SD component can be parameterized as in eq. (3.3) in terms of the effective couplings F_V , $g_{P\gamma}^V$, and $g_{PV'}^V$. Here one could use the reference values in table 1 as starting point of the fit, and eventually improve the determination of some of these couplings (especially by means of $|\pi^0\pi^0\gamma\rangle$ data, where the vector component is dominant).
- For the scalar SD component we propose a parameterization of type (4.7) with a resonant part (with or without the $f_0(600)$ pole) and a polynomial non-resonant term (with one or two free parameters for each channel). In the two $|\pi\pi\gamma\rangle$ channels the parameterization of the non-resonant part can be improved with the inclusion of appropriate re-scattering phases, as shown in eq. (4.14). These do not involve additional free parameters since the β_i in eq. (4.14) can be determined using the precise low-energy constraints on $\pi\pi$ phase shifts, as outlined in section 4.1. An important consistency check of the whole approach is obtained by the combined fit of the two $|\pi\pi\gamma\rangle$ channels, which should satisfy the isospin decomposition (4.9).

Recently, this procedure has been employed in the analysis of KLOE data on $e^+e^- \rightarrow \pi^+\pi^-\gamma$ [12] with satisfactory results. A more significant test of the method should be possible in the near future, with the combined analysis of high-statistics data on both $e^+e^- \rightarrow \pi^+\pi^-\gamma$ and $e^+e^- \rightarrow \pi^0\pi^0\gamma$.

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